Math 62: 8.8 Variation
Sullivan-Struve-Mazzarella
also work rates - preparation for final exam

Math 72: 6.6 Variation
Rockswold

Objectives

1) Solve applications using variation

a) Direct variation = Directly proportional

b) Inverse variation

c) Joint variation

CAUTION:

While direct variation problems can be solved correctly using proportions, inverse and joint variation problems cannot be solved correctly using proportions.

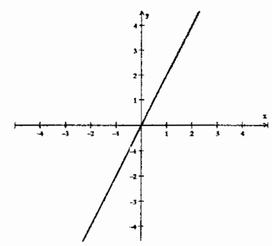
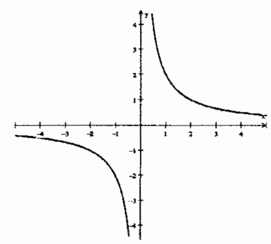
Direct variation problems can be solved using the constant of variation and equation of variation method which is used for inverse or joint variation.

Math 70 7.8 Variation

Objectives:

1) Direct variation
2) Inverse variation
3) Joint variation

Key words for recognizing variation problems: "varies" or "proportional"

	Model Words	**Translation to math**	**Model equation**	**Graph of model if $k=2$**
Direct	"y varies directly as x" "y is directly proportional to x" "y varies as x"	"y varies" is always $y = k \cdot$ "directly" means x is in the numerator of RHS	$y = kx$	
Inverse	"y varies inversely as x" "y is inversely proportional to x"	"y varies" is always $y = k \cdot$ "inversely" means x is in the denominator of RHS	$y = k \cdot \frac{1}{x}$ or $y = \frac{k}{x}$	
Joint	"Joint" means that 3 or more variables are involved. <u>Example</u>: "y varies jointly as x and the square of w, and inversely as z and the cube root of v"	"y varies" is always $y = k \cdot$ "directly" means x and w^2 are in the numerator of RHS "inversely" means z and $\sqrt[3]{w}$ are in the denominator of RHS	$y = k \cdot \frac{x \cdot w^2}{z \cdot \sqrt[3]{v}}$	Not a 2-dimensional graph

<u>Cautions</u>:

- Direct variation problems can be solved by proportions. Inverse and Joint variation cannot!
- Variation equations have only multiply and divide, never add or subtract.
- Always write units on the final answer.
- Use units to help identify which values go with which variables.

Process that works for ANY type of variation problem:

<u>Step 1:</u> Define any variables, if needed. (Use letters that make sense.)
<u>Step 2:</u> Translate the sentence into an equation of variation. Don't forget *k*!
<u>Step 3:</u> Substitute a complete set of numbers (given in question) and solve for *k*.
<u>Step 4:</u> Solve the incomplete set of numbers (given in question) and solve to answer the question.

Examples

1. Hooke's law states that the distance a spring stretches is directly proportional to the weight attached to the spring. If a 40-pound weight attached to a spring stretches the spring 5 inches, find the distance that a 65 pound weight will stretch that same spring.

2. Boyle's law says that if the temperature stays the same, the pressure P of a gas is inversely proportional to the volume V. If a cylinder in a steam engine has a pressure of 960 kilopascals when the volume is 1.4 cubic meters, find the pressure when the volume increases to 2.5 cubic meters.

3. The lateral surface area of a cylinder varies jointly as its radius and height.
   a. Express this surface area S in terms of radius r and height h.
   b. If the lateral surface area is 20π square cm when the radius is 2 cm and the height is 5 cm, find the exact constant of variation and the equation of variation.
   c. Find the radius when the lateral surface area is 40π square cm and the height is 2 cm.

4. The maximum weight that a circular column can support is directly proportional to the fourth power of its diameter and is inversely proportional to the square of its height. A 2-meter-diameter column that is 8 meters in height can support 1 ton. Find the weight that a 1-meter-diameter column that is 4 meters in height can support.

Extras:

1. The maximum weight that a rectangular beam can support varies jointly as its width and the square of its height and inversely as its length. If a beam ½ foot wide, 1/3 foot high, and 10 feet long can support 12 tons, find how much a similar beam can support if the beam is 2/3 foot wide, ½ foot high, and 16 feet long.

2. The horsepower to drive a boat varies directly as the cube of the speed of the boat. If the speed of the boat is to double, determine the corresponding increase in horsepower required.

3. The volume of a cone varies jointly as its height and the square of its radius. If the volume of a cone is 32π cubic inches when the radius is 4 inches and the height is 6 inches, find the volume of a cone when the radius is 3 inches and the height is 5 inches.

4. The intensity of light (in foot-candles) varies inversely as the square of x, the distance in feet from the light source. The intensity of light 2 feet from the source is 80 foot-candles. How far away is the source if the intensity of light is 5 foot-candles?

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Variation and Problem Solving

1) Direct variation.
2) Inverse variation } Do not solve by proportions!
3) Joint variation

key words for recognizing these problems are
* varies
* proportional.

However, "proportional" suggests that these problems should be solved using proportions, but only the first type can be solved using ordinary proportions.

<u>Goal</u>: one method for all 3 types of variation

First step: Translate each type of variation to an equation of variation

Direct Variation

model problem: y varies directly as x
or: y is directly proportional to x

mean $y = k \cdot x$ ← x in numerator of RHS

As x increases, y increases

k is a number, a constant, called the constant of variation.
All variation problems have k.
k is essential. The problem can't be done without it.

Inverse Variation

model problem: y varies inversely as x
or. y is inversely proportional to x.

mean $y = k \cdot \frac{1}{x}$ ← x in denominator of RHS

or $y = \frac{k}{x}$

As x increases, y decreases

Joint variation means

- 3 or more variables involved (not including k, which is not a variable)
- Each variable is either direct (in numerator) or inverse (in denominator).
- If problem does not say "direct" or "inverse", assume it's direct and write it in the numerator.

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Examples of joint variation equations.

① y varies jointly as x and z

means $\boxed{y = k \cdot x \cdot z}$

② y varies jointly as x and inversely as z

means $\boxed{y = k \cdot \frac{x}{z}}$

③ P varies jointly as q and the square of r

means $\boxed{P = \frac{k \cdot q}{r^2}}$

Process for Solving Variation Problems

step 1: Translate to an equation of variation, with k.

step 2: Solve for k.
In the problem, you will be given a complete set of numbers, one for each variable.
Plug them all in, and solve the result for k.

* Once we know the value of k, we can use it for all of the rest of the problem. *

step 3: Answer the question.
In the problem, you will be given a partial set of numbers, one for every variable except the question.
Plug them all in, plug in the value of k from step 2, and solve for the requested variable.

Helpful notes:

- Variation equations have only multiply and divide. There is never add or subtract.
- Always write units on answers.
- Use units to help identify which values are which variables.
- Use letters that remind you of their meanings, especially in joint problems with many variables.

Math 70 Practice Problems for Variation

1. Hooke's law states that the distance a spring stretches is directly proportional to the weight attached to the spring. If a 40-pound weight attached to a spring stretches the spring 5 inches, find the distance that a 65 pound weight will stretch that same spring.

step 1: $D = k \cdot W$ D = distance, W = weight

step2: $5 = k \cdot 40$

$\frac{1}{8} = k$

step3: $D = \frac{1}{8} \cdot 65 = \boxed{\frac{65}{8}\text{ cm}} = \boxed{8.125\text{cm}}$

2. Boyle's law says that if the temperature stays the same, the pressure P of a gas is inversely proportional to the volume V. If a cylinder in a steam engine has a pressure of 960 kilopascals when the volume is 1.4 cubic meters, find the pressure when the volume increases to 2.5 cubic meters.

step1: $P = k \cdot \frac{1}{V}$

step2: $960 = \frac{k}{1.4}$

$1344 = k$

step3: $P = \frac{1344}{2.5}$

$P = \boxed{537.6 \text{ kiloPascals}}$

3. The lateral surface area of a cylinder varies jointly as its radius and height.
   a. Express this surface area S in terms of radius r and height h.
   b. If the lateral surface area is 20π square cm when the radius is 2 cm and the height is 5 cm, find the exact constant of variation and the equation of variation.
   c. Find the radius when the lateral surface area is 40π square cm and the height is 2 cm.

a = step1: $\boxed{S = k \cdot r \cdot h}$

b = step2: $20\pi = k \cdot 2 \cdot 5$

$\boxed{2\pi = k}$

rewrite eqn $\boxed{S = 2\pi r h}$

c = step3: $40\pi = 2\pi \cdot r \cdot 2$

$\boxed{10\text{cm} = r}$

4. The maximum weight that a circular column can support is directly proportional to the fourth power of its diameter and is inversely proportional to the square of its height. A 2-meter-diameter column that is 8 meters in height can support 1 ton. Find the weight that a 1-meter-diameter column that is 4 meters in height can support.

max weight column supports   is directly proportional   4th power diameter, inversely square of height   weight = w, diameter = d, height = h

step1: $w = k \cdot \frac{d^4}{h^2}$

step2: $\left.\begin{matrix} d=2 \\ h=8 \\ w=1 \end{matrix}\right\} \Rightarrow 1 = k \cdot \frac{2^4}{8^2} \Rightarrow 1 = \frac{16k}{64} \Rightarrow 64 = 16k \Rightarrow k = 4$

step 3: $\left.\begin{matrix} w=? \\ d=1 \\ h=4 \end{matrix}\right\} \Rightarrow W = \frac{4 \cdot 1^4}{4^2} \Rightarrow \boxed{W = \frac{1}{4}\text{ ton}}$

Extras:

1. The maximum weight that a rectangular beam can support varies jointly as its width and the square of its height and inversely as its length. If a beam ½ foot wide, 1/3 foot high, and 10 feet long can support 12 tons, find how much a similar beam can support if the beam is 2/3 foot wide, ½ foot high, and 16 feet long.

M = max weight
w = width
h = height
l = length

step1 max weight rectangular beam — vary jointly — -width -square height -inversely length

$M = \frac{k \cdot w \cdot h^2}{l}$

step2: $w = \frac{1}{2}$, $h = \frac{1}{3}$, $l = 10$, $M = 12$ $\Rightarrow 12 = \frac{k \cdot (\frac{1}{2})(\frac{1}{3})^2}{10} \Rightarrow 120 = \frac{1}{18}k \Rightarrow k = 2160$

step3: $w = \frac{2}{3}$, $h = \frac{1}{2}$, $l = 16$, $M = ?$ $\Rightarrow M = \frac{2160 \cdot (\frac{2}{3}) \cdot (\frac{1}{2})^2}{16}$

$\boxed{M = 22.5 \text{ tons}}$

2. The horsepower to drive a boat varies directly as the cube of the speed of the boat. If the speed of the boat is to double, determine the corresponding increase in horsepower required.

step1: horsepower — varies directly — cube of speed

H = horsepower
s = speed

$H = k \cdot s^3$

step2: double the speed ⇒ (2s) substitute and simplify

$H = k(2s)^3$

$H = k \cdot 8 \cdot s^3$

$H = 8k \cdot s^3$

$\boxed{\text{horsepower must be multiplied by 8}}$

3. The volume of a cone varies jointly as its height and the square of its radius. If the volume of a cone is 32π cubic inches when the radius is 4 inches and the height is 6 inches, find the volume of a cone when the radius is 3 inches and the height is 5 inches.

V = volume
h = height
r = radius

step1: volume cone — varies jointly — height square radius

$V = k \cdot hr^2$

step2: $V = 32\pi$, $r = 4$, $h = 6$ $\Rightarrow 32\pi = k \cdot 6 \cdot 4^2 \Rightarrow 32\pi = 96k \Rightarrow k = \frac{32\pi}{96} = \frac{\pi}{3}$

step3: $V = ?$, $r = 3$, $h = 5$ $\Rightarrow V = \frac{\pi}{3} \cdot 5 \cdot 3^2 \Rightarrow$ $\boxed{V = 15\pi \text{ cubic inches}}$ or $\boxed{15\pi \text{ in}^3}$

4. The intensity of light (in foot-candles) varies inversely as the square of x, the distance in feet from the light source. The intensity of light 2 feet from the source is 80 foot-candles. How far away is the source if the intensity of light is 5 foot-candles?

I = intensity in foot-candles
x = distance in feet

step1: Intensity — varies inversely — square of x = distance

$I = \frac{k}{x^2}$

step2: $I = 80$ ft-candles, $x = 2$ ft $\Rightarrow 80 = \frac{k}{2^2} \Rightarrow 80 = \frac{k}{4} \Rightarrow k = 320.$

step3: $x = ?$, $I = 5$ $\Rightarrow 5 = \frac{320}{x^2} \Rightarrow 5x^2 = 320 \Rightarrow x^2 = \frac{320}{5} \Rightarrow x^2 = 64$

distance X cannot be negative $\Rightarrow x = \pm\sqrt{64} \rightarrow$ $\boxed{x = 8 \text{ ft}}$

Math 62 **Work Rates**

Objectives

1) Work rates

Examples

Solve.

1) A painter can finish painting house in 5 hours. Her assistant takes 7 hours to finish the same job. How long would it take for them to complete the job if they were working together?

2) One pump can drain a pool in 9 minutes. When a second pump is also used, the pool only takes 4 minutes to drain. How long would it take the second pump to drain the pool if it were the only pump in use?
3) One conveyor belt can move 1000 boxes in 7 minutes. Another can move 1000 boxes in 10 minutes. If another conveyor belt is added and all three are used, the boxes are moved in 3 minutes. How long would it take the third conveyor belt along to do the same job?

4) A baker can decorate the day's cookie supply four times as fast as his new assistant. It takes 16 minutes for them to decorate the day's cookie supply if they work together. How long does each one take if working alone?

5) Mark and Rachel both work for Smith Landscaping Company. Mark can finish a planting job in 2 hours, while if takes Rachel hours 4 hours to finish the same job. If Mark and Rachel will work together on the job, and the cost of labor is $40 per hour, what should the labor estimate be? (Round to the nearest cent, if necessary.)

Work rates

① If the painter takes 5 hours to do the entire job, the painter can do $1/5$ of the job in one hour, or .2 job/hour. This is the painter's work rate.
The assistant's work rate is $1/7$ job per hour.

Method 1: The unknown work rate is based on the unknown time, x hours to complete the job together. This makes that work rate $1/x$, or the fraction they do in one hour.

Equation: add fractions done alone in one hour to get fraction done together in one hour

$$\frac{1}{5}+\frac{1}{7}=\frac{1}{x}$$

$$35x\cdot\frac{1}{5}+35x\cdot\frac{1}{7}=35x\cdot\frac{1}{x}$$ multiply by LCD $= 35x$

$$7x+5x=35$$ cancel $\frac{35x}{5}=7x$, $\frac{35x}{7}=5x$, $\frac{35\cancel{x}}{\cancel{x}}=35$

$$12x=35$$

$$\boxed{x=\frac{35}{12}\text{ hrs}}$$

$$\boxed{x=2.91\overline{6}\text{ hours}}$$

Because working together means working more quickly, the time together should be smaller than either time alone.

Method 2: Multiply each work rate by the time worked together to get work done by each. Add to get one (1) job.

$$\frac{x}{5}+\frac{x}{7}=1$$ multiply by LCD $=35$

$$35\cdot\frac{x}{5}+35\cdot\frac{x}{7}=35\cdot 1$$ cancel

$$7x+5x=35$$

Continue as in Method 1.

② one pump 9 min $\Rightarrow$ $\frac{1}{9}$ of job alone

both ("also") 4 min $\Rightarrow$ $\frac{1}{4}$ of job together

second pump x min $\Rightarrow$ $\frac{1}{x}$ of job alone

Method 1: $\frac{1}{9} + \frac{1}{x} = \frac{1}{4}$ LCD $= 36x$

$$4x + 36 = 9x$$

$$36 = 5x$$

$$7.2 = \frac{36}{5} = x$$

$\boxed{7.2 \text{ min}}$

Method 2: $\frac{4}{9} + \frac{4}{x} = 1$ LCD $= 9x$

$4x + 36 = 9x$ con't as before.

③ one belt 7 min $\Rightarrow$ $\frac{1}{7}$ of job alone

another belt 10 min $\Rightarrow$ $\frac{1}{10}$ of job. alone

3rd belt, all together 3 min $\Rightarrow$ $\frac{1}{3}$ of job together

x = 3rd belt time alone $\Rightarrow$ $\frac{1}{x}$ of job alone

Method 1: $\frac{1}{7} + \frac{1}{10} + \frac{1}{x} = \frac{1}{3}$ LCD $= 210x$

$$30x + 21x + 210 = 70x$$

$$51x + 210 = 70x$$

$$210 = 19x$$

$$\frac{210}{19} = x$$

$\boxed{\frac{210}{19} \text{ min}}$

check
$\frac{210}{19} \approx 11.05$ min.

Method 2: $\frac{3}{7} + \frac{3}{10} + \frac{3}{x} = 1$ LCD $= 70x$

$30x + 21x + 210 = 70x$ con't as before

(4) Baker 4x as fast $\Rightarrow$ $\frac{1}{4}$ the time!

x = time = time for baker $\Rightarrow$ $1/x$

$4x$ = bigger time = time for assistant $\Rightarrow$ $1/4x$

16 = time together $\Rightarrow$ $1/16$

Method 1

$$\frac{1}{x} + \frac{1}{4x} = \frac{1}{16} \qquad LCD = 16x$$

$$16 + 4 = x$$

$$\boxed{20 \text{ min} = x = \text{baker}}$$

$$4x(20) = \boxed{80 \text{ min} = \text{assistant}}$$

Method 2:

$$\frac{16}{x} + \frac{16}{4x} = 1 \qquad LCD = 4x$$

$$64 + 16 = 4x$$

$$80 = 4x$$

$$20 = \frac{80}{4} = x$$

$$\boxed{\begin{array}{l} x = 20 \text{ min Baker} \\ 4x = 80 \text{ min assistant} \end{array}}$$

(5) Mark – 2 hrs

Rachel – 4 hrs.

Together – x hrs.

\$40 per hour (per person!)

Method 1

$$\frac{1}{2} + \frac{1}{4} = \frac{1}{x} \qquad LCD = 4x$$

$$2x + x = 4$$

$$3x = 4$$

$$x = 4/3 \text{ hr.}$$

Method 2:

$$\frac{x}{2} + \frac{x}{4} = 1 \qquad LCD = 4$$

$2x + x = 4$ con't as before

$$4/3 \text{ hr} \times \$40/\text{person} \times 2 \text{ people}$$

$$= 106\tfrac{2}{3} = \boxed{\$106.67}$$